arXiv:1101.2119v3 [math.FA] 14 Feb 2011

SOME REMARKS ON ABSOLUTELY SUMMING MULTILINEAR OPERATORS

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Abstract. This short note has a twofold purpose:

(i) to solve the question that motivates a recent paper of D. Popa on multilinear variants of Pietsch's composition theorem for absolutely summing operators. More precisely, we remark that there is a natural perfect extension of Pietsch's composition theorem to the multilinear and polynomial settings. This fact was overlooked in the aforementioned paper;

(ii) to investigate extensions of some results of the aforementioned paper for particular situations, mainly by exploring cotype properties of the spaces involved.

When dealing with (ii) we also prove an useful, albeit simple, result of independent interest (which is a consequence of recent arguments used in a recent paper of O. Blasco *et al.*). The result asserts that if X_1 has cotype 2 and $1 \leq p \leq s \leq 2$, then every absolutely $(s; s, t, ..., t)$-summing multilinear operator from $X_1 \times \cdots \times X_n$ to Z is absolutely $(p; p, t, ..., t)$-summing, for all $t \geq 1$ and all $X_2, ..., X_n, Z$. In particular, under the same hypotheses, every absolutely $(s; s, ..., s)$-summing multilinear operator from $X_1 \times \cdots \times X_n$ to Z is absolutely $(p; p, ..., p)$-summing. A similar result holds when X_1 has cotype greater than 2 (and obviously, *mutatis mutandis,* when X_1 is replaced by X_j with $j \neq 1$). These results generalize previous results of H. Junek *et al.*, G. Botelho *et al.* and D. Popa.

In the last section we show that a straightforward argument solves partially another problem from the aforementioned paper of D. Popa.

1. Introduction

In this note the letters $X_1, ..., X_n, X, Y, Z$ will always denote Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and X^* represents the topological dual of X.

The concept of absolutely p-summing linear operators is due to A. Pietsch [32]. If $1 \leq q \leq p < \infty$, we say that a continuous linear operator $u : X \to Y$ is absolutely $(p; q)$-summing if $(u(x_j))_{j=1}^{\infty} \in \ell_p(Y)$ whenever $(x_j)_{j=1}^{\infty} \in \ell_q^w(X)$, where $\ell_q^w(X) := \{(x_j)_{j=1}^{\infty} \subset X : \sup_{\varphi \in B_{X^*}} \sum_j |\varphi(x_j)|^q < \infty\}$.

The class of absolutely $(p; q)$-summing linear operators from X to Y will be represented by $\Pi_{p,q}(X; Y)$ and by $\Pi_p(X; Y)$ if $p = q$. From now on the space of all continuous n-linear operators from $X_1 \times \cdots \times X_n$ to Y will be denoted by $\mathcal{L}(X_1, ..., X_n; Y)$.

If $0 < p, q_1, ..., q_n < \infty$ and $\frac{1}{p} \leq \frac{1}{q_1} + \cdots + \frac{1}{q_n}$, a multilinear operator $T \in \mathcal{L}(X_1, ..., X_n; Y)$ is absolutely $(p; q_1, ..., q_n)$-summing if $(T(x_j^{(1)}, ..., x_j^{(n)}))_{j=1}^{\infty} \in \ell_p(Y)$ for every $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{q_k}^w(X_k), k = 1, ..., n$. In this case we write $T \in \Pi_{p;q_1,...,q_n}^n(X_1, ..., X_n; Y)$. If $q_1 = \cdots = q_n = q$, we sometimes write $\Pi_{p;q}^n(X_1, ..., X_n; Y)$ instead of $\Pi_{p;q,...,q}^n(X_1, ..., X_n; Y)$ and if $q_1 = \cdots = q_n = q = p$ we simply write $\Pi_p^n(X_1, ..., X_n; Y)$ instead of $\Pi_{p;p}^n(X_1, ..., X_n; Y)$. In

2010 Mathematics Subject Classification: Primary 46G25; Secondary 47L22, 47H60.

Key words: absolutely p-summing multilinear operators, cotype.

the special case in which $p = q/n$ this class has special properties and the operators in $\Pi^n_{\frac{q}{n};q}$ are called q-dominated operators. Here we will use the notation $\delta^n_q = \Pi^n_{\frac{q}{n};q}$.

Finally, we recall the class of multiple $(p;q)$-summing multilinear operators. If $1 \leq q \leq p < \infty$, a multilinear operator $T \in \mathcal{L}(X_1, ..., X_n; Y)$ is multiple $(p;q)$-summing if $(T(x^{(1)}_{j_1}, ..., x^{(n)}_{j_n}))^{\infty}_{j_1,..,j_n=1} \in \ell_p(Y)$ for every $(x^{(k)}_j)^{\infty}_{j=1} \in \ell^w_{q_k}(X_k), k = 1, ..., n$. In this case we write $T \in \Pi^{\text{mult},n}_{p;q}(X_1, ..., X_n; Y)$ or $\Pi^{\text{mult},n}_{p}(X_1, ..., X_n; Y)$ if $p = q$.

For details on the linear theory of absolutely summing operators we refer to the classical monograph [17] and for recent developments we refer to [9, 12, 22, 23, 37] and references therein; for the multilinear theory we refer, for example, to [13, 30, 34] and references therein.

One important result of the linear theory of absolutely summing linear operators is Pietsch's composition theorem:

If $p, q \in (1, \infty)$ and $r \in [1, \infty)$ are such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then

$$\Pi_q \circ \Pi_p \subset \Pi_r. \tag{1.1}$$

In a recent paper [36] this result is investigated in the context of multilinear mappings. The first question faced in [36] was to decide what should be the natural class of absolutely p-summing n-linear mappings $\mathcal{I}^n_p$ such that the analogous result would hold in the multilinear setting. More precisely the following problem summarize mathematically the motivation of the paper [36] (see [36, Section 1]):

Problem 1.1. *If $p, q \in (1, \infty)$ and $r \in [1, \infty)$ are such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, does the inclusion*

$$\Pi_q \circ \mathcal{I}^n_p \subset \mathcal{I}^n_r \tag{1.2}$$

always hold for all natural numbers n and some natural n-linear extension $(\mathcal{I}^n_s)^{\infty}_{s=1}$ of $(\Pi_s)^{\infty}_{s=1}$?

In [36] it is shown that the inclusion (1.2) is far from being true for the class of dominated n-linear mappings, i.e., $\mathcal{I}^n_p = \delta^n_p$ and $\mathcal{I}^n_r = \delta^n_r$. So the author decided to investigate the case $\mathcal{I}^n_p = \delta^n_p$ and $\mathcal{I}^n_r = \Pi^n_r$, i.e., the following question was considered:

Problem 1.2. *Let $p, q \in (1, \infty)$ and $r \in [1, \infty)$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. For what natural numbers n the inclusion*

$$\Pi_q \circ \delta^n_p \subset \Pi^n_r$$

is true?

Among other interesting results, in [36, Theorem 4 and Corollary 19] it is proved that the above inclusion is valid for all n and $r \in [1, 2]$:

Theorem 1.3. *([36]) Let $p, q \in (1, \infty)$ and $r \in [1, 2]$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then*

$$\Pi_q \circ \delta^n_p \subset \Pi^n_r \tag{1.3}$$

for all natural numbers n.

In view of Theorem 1.3 the following problem is posed [36] (in the last section we use a very simple remark to solve this problem for all $n \geq \frac{p}{r}$):

Problem 1.4. *Let $p, q \in (1, \infty)$ and $r \in (2, \infty)$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. For what natural numbers n the inclusion $\Pi_q \circ \delta_p^n \subset \Pi_r^n$ is true?*

We believe that by considering different classes (p-dominated and absolutely r-summing n-linear operators), the Problem 1.2 becomes a little bit far from the original motivation (1.1). But, of course, Problem 1.2 has its intrinsic mathematical interest and a complete solution seems to be far from being simple.

It is worth mentioning that the class Π_r^n (although this class had been broadly explored by several authors and also offers interesting challenging problems) is usually not considered as a completely adequate extension of Π_r, since several of the linear properties of Π_r are not lifted to Π_r^n (this kind of fault of the class Π_r^n - and its polynomial version - was discussed in some recent papers (see, for example, [35, page 167] and [14, 15, 28])). Using the terminology of [14] it can be said that the ideal of absolutely r-summing n-homogeneous polynomials (associated to Π_r^n) is not compatible with the linear operator ideal Π_r. For details on operator ideals we refer to the classical monograph [33] and [18].

The case $r = 1$ of Theorem 1.3 ([36, Theorem 4]) deserves some special attention. Contrary to the case $n = 1$, for $n \geq 2$, in many cases, i.e., for several Banach spaces $X_1, ..., X_n, Y$, the space $\Pi_1^n(X_1, ..., X_n; Y)$ coincides with the whole space of continuous multilinear operators $\mathcal{L}(X_1, ..., X_n; Y)$ and Theorem 1.3 (with $r = 1$) becomes useless. For example:

- For all Banach spaces $X_1, ..., X_n$ the folkloric Defant-Voigt Theorem asserts that

$$\Pi_1^n(X_1, ..., X_n; \mathbb{K}) = \mathcal{L}(X_1, ..., X_n; \mathbb{K}). \tag{1.4}$$

- ([3]) If each X_j is a Banach space with cotype q_j for every j and $1 \leq \frac{1}{q_1} + \cdots + \frac{1}{q_n}$, then

$$\Pi_1^n(X_1, ..., X_n; Y) = \mathcal{L}(X_1, ..., X_n; Y) \tag{1.5}$$

  for every Banach space Y.

It must be said that the paper [36] also presents several interesting variants of Theorem 1.3 (including the case $r = 1$), replacing, for example, Π_1^n by $\Pi_{t;r}^n$ for some values of t, r (depending on n, in general).

This short note has two main goals. The first goal is to present the precise ideal $\mathcal{I}_p^n$ that solves completely Problem 1.1. Our second goal is to look for stronger variants of Theorem 1.3, specially under certain special cotype assumptions. For example (using a completely different approach from the one in [36]), we show that when X_j has cotype 2 for some j and Y has cotype 2 then the inclusion

$$\Pi_q(Y, Z) \circ \delta_p^n(X_1, ..., X_n; Y) \subset \Pi_r^n(X_1, ..., X_n; Z)$$

from Theorem 1.3 (with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $r \in [1, 2]$) can be replaced by

$$\Pi_q(Y; Z) \circ \bigcup\nolimits_{p \geq 1} \delta_p^n(X_1, ..., X_n; Y) \subset \Pi_r^n(X_1, ..., X_n; Z)$$

for all $q \in [1, \infty)$, all $r \in [1, 2]$ and all Banach spaces $X_1, ...X_{j-1}, X_{j+1}, ..., X_n, Z$.

In the last section we give a simple partial answer to Problem 1.4 by showing that the inclusion holds whenever $n \geq \frac{p}{r}$ (in fact we do not need that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. This fact is apparently overlooked in [36]).

2. The solution to Problem 1.1

The ideal of absolutely p-summing linear operators has various possible generalizations to multi-ideals: absolutely p-summing multilinear operators ([1, 26]), p-dominated multilinear operators ([3, 16, 24, 30]), strongly p-summing multilinear operators ([19]), strongly fully p-summing multilinear operators ([8]), multiple p-summing multilinear operators ([25, 31]), absolutely p-summing multilinear operators by the method of linearization ([4]), p-semi-integral multilinear operators ([13]) and the composition ideal generated by the ideal of absolutely p-summing linear operators ([10]).

Each of these classes has its own properties and shares part of the spirit of the linear concept of absolutely p-summing operators. The richness of the multilinear theory of absolutely summing operators and multiplicity of different possible approaches has attracted the attention of several mathematicians in the last two decades. One of the beautiful features is that no one of these classes shares all the desired properties of the ideal of absolutely p-summing linear operators and depending on the properties that one looks for, the "natural" class to be considered changes. However, it seems to be clear that the most popular classes until now are the ideals of p-dominated multilinear operators and multiple p-summing multilinear operators (but the classes that seem to be closest to the essence of the linear ideal are, in our opinion, the classes of strongly p-summing multilinear operators and strongly fully (also called strongly multiple) p-summing multilinear operators). For a recent survey on this subject we refer to [28].

In this section we remark that the composition ideal generated by the absolutely p-summing multilinear operators is precisely the class that completely answers Problem 1.1.

If $\mathcal{I}$ is an operator ideal it is always possible to consider the class

$$\mathcal{C}_{\mathcal{I}}^{n} := \{u \circ A : A \in \mathcal{L}^{n} \text{ and } u \in \mathcal{I}\},$$

where $\mathcal{L}^n$ denotes the class of all continuous n-linear operators between Banach spaces. So, for Banach spaces $X_1, ..., X_n, Y, Z$, an n-linear operator $T : X_1 \times \cdots \times X_n \to Y$ belongs to $\mathcal{C}_{\mathcal{I}}^{n}(X_1, ..., X_n; Y)$ if and only if there are a Banach space Z, a map $A \in \mathcal{L}(X_1, ..., X_n; Z)$ and $v \in \mathcal{I}(Z; Y)$ so that

$$T(x_1, ..., x_n) = v\,(A(x_1, ..., x_n)).$$

It is well known that $\mathcal{C}_{\mathcal{I}}^{n}$ is an ideal of n-linear mappings (for details see [10]). The case where $\mathcal{I} = \Pi_1$ was investigated in [10], where it was shown that this class lifts, to the multilinear setting, various important features of the linear ideal, such as a Dvoretzky-Rogers theorem, a Grothendieck theorem and a Lindenstrauss-Pełczyński theorem (three important cornerstones of the linear theory of absolutely summing operators). The solution to Problem 1.1 is now quite simple and the proof is a straightforward consequence of Pietsch's composition theorem for absolutely summing linear operators:

Proposition 2.1. *If $p,q\in(1,\infty)$ and $r\in[1,\infty)$ are such that $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$, then*

$$\Pi_q\circ\mathcal{C}^n_{\Pi_p}\subset\mathcal{C}^n_{\Pi_r}$$

for all natural number n.

Remark 2.2. *It is worth mentioning that the polynomial version of the multi-ideal $(\mathcal{C}^n_{\mathcal{I}})_{n=1}^{\infty}$ (which we denote by $(\mathcal{CP}^n_{\mathcal{I}})_{n=1}^{\infty}$) also solves the polynomial version of Problem 1.1. Moreover, the polynomial ideal $(\mathcal{CP}^n_{\mathcal{I}})_{n=1}^{\infty}$ is a coherent sequence and compatible with $\mathcal{I}$ (see [14]), reinforcing that the composition method is an adequate method for generalizing the ideal of absolutely summing operators.*

3. Some remarks related to Problem 1.2

Although the very simple solution to Problem 1.1, we do think that Problem 1.2 is interesting and now we investigate how the results from [36] can be improved in certain special situations. In view of the intuitive "small size" (in general) of the class δ_p^n (see [6, 7, 11] for details that justify this intuition), in this section we look for results of the type

$$\Pi_q\circ\bigcup_{p\geq1}\delta_p^n\subset\Pi^n_{r;s}$$

for $q,r,s\in[1,\infty)$, i.e., for stronger results than those proposed in the Problem 1.2. More precisely, for fixed Banach spaces $X_1,...,X_n,Y,Z$ with certain properties we try to find t,l,r,s so that

$$\Pi_{t;l}(Y;Z)\circ\bigcup_{p\geq1}\delta_p^n(X_1,...,X_n;Y)\subset\Pi^n_{r;s}(X_1,...,X_n;Z).$$

In view of the important effect that cotype properties play in the theory of absolutely summing operators (see for example [5, 12, 27, 38]), in the next section we, in some sense, complement the results of [36] by exploring the cotype of the Banach spaces involved.

If $Y=\mathbb{K}$ the following result gives an important and useful estimate for the "size" of the set of all p-dominated scalar-valued multilinear operators:

Theorem 3.1 (Floret, Matos (1995) and Pérez-García (2002))**.** *If $X_1,...,X_n$ are Banach spaces then*

$$\bigcup_{p\geq1}\delta_p^n(X_1,...,X_n;\mathbb{K})\subset\Pi^n_{(1;2,...,2)}(X_1,...,X_n;\mathbb{K}).$$

More precisely this result is due to Floret-Matos [20] for the complex case and due to D. Pérez-García [29] for the general case. It is worth mentioning that, besides not explicitly mentioned, this result seems to be essentially re-proved in [36].

The following result is an application of the previous theorem:

Proposition 3.2. *If $X_1,...,X_n,Y,Z$ are Banach spaces, then*

$$\Pi_{s;1}(Y;Z)\circ\bigcup_{p\geq1}\delta_p^n(X_1,...,X_n;Y)\subset\Pi^n_{(s;2,...,2)}(X_1,...,X_n;Z).$$

for all $s\geq1$.

Proof. Let $T \in \Pi_{(s;1)}(Y,Z)$ and $R \in \bigcup_{p\geq 1}\delta_p^n(X_1,...,X_n;Y)$. Consider $(x_j^{(k)})_{j=1}^{\infty} \in \ell_2^w\left(X_k\right)$ for all $k=1,...,n$. If $\varphi \in Y^*$ from Theorem 3.1 we have

$$\varphi \circ R \in \bigcup_{p\geq 1}\delta_p^n(X_1,...,X_n;\mathbb{K}) \subset \Pi^n_{(1;2,...,2)}(X_1,...,X_n;\mathbb{K}).$$

Hence

$$\left(\varphi\left(R(x_j^{(1)},...,x_j^{(n)})\right)\right)_{j=1}^{\infty} \in \ell_1.$$

We thus conclude that $\left(R(x_j^{(1)},...,x_j^{(n)})\right)_{j=1}^{\infty} \in \ell_1^w\left(Y\right)$ and so

$$\left(T\left(R(x_j^{(1)},...,x_j^{(n)})\right)\right)_{j=1}^{\infty} \in \ell_s\left(Z\right),$$

because $T \in \Pi_{s;1}(Y,Z)$. □

When $X_1 = \cdots = X_n$ are $\mathcal{L}_\infty$ spaces we have a quite stronger result:

Proposition 3.3. *If Y,Z are Banach spaces and $X_1 = \cdots = X_n$ are $\mathcal{L}_\infty$ spaces, then*

$$\Pi_{(s;r)}(Y;Z) \circ \mathcal{L}(X_1,...,X_n;Y) \subset \Pi^n_{(s;2r,...,2r)}(X_1,...,X_n;Z)$$

for all $s \geq r \geq 1$.

Proof. Let $T \in \Pi_{(s;r)}(Y,Z)$ and $R \in \mathcal{L}(X_1,...,X_n;Y)$. Consider $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{2r}^w\left(X_k\right)$ for all $k=1,...,n$. If $\varphi \in Y^*$ we have (from [5, Theorem 3.15])

$$\varphi \circ R \in \mathcal{L}(X_1,...,X_n;\mathbb{K}) = \Pi^n_{(r;2r,...,2r)}(X_1,...,X_n;\mathbb{K}).$$

Hence

$$\left(\varphi\left(R(x_j^{(1)},...,x_j^{(n)})\right)\right)_{j=1}^{\infty} \in \ell_r$$

and thus $\left(R(x_j^{(1)},...,x_j^{(n)})\right)_{j=1}^{\infty} \in \ell_r^w\left(Y\right)$. Since $T \in \Pi_{(s;r)}(Y,Z)$, we conclude that

$$\left(T\left(R(x_j^{(1)},...,x_j^{(n)})\right)\right)_{j=1}^{\infty} \in \ell_s\left(Z\right).$$

□

4. Exploring the cotype of the spaces involved

In this section we will need, as auxiliary results, some inclusions involving cotype and absolutely summing multilinear operators. The following results can be found in [21, Theorem 3 and Remark 2] and [5, Theorem 3.8], by using complex interpolation, and [35, Corollary 4.6]:

Theorem 4.1 (Inclusion Theorem)**.** *Let $X_1,...,X_n$ be Banach spaces with cotype s and $n \geq 2$ be a positive integer:*
(i) If $s=2$, then

$$\Pi^n_{q;q}(X_1,...,X_n;Y) \subset \Pi^n_{p;p}(X_1,...,X_n;Y)$$

holds true for $1 \leq p \leq q \leq 2$.

(ii) If $s > 2$, then

$$\Pi^n_{q;q}(X_1, ..., X_n; Y) \subset \Pi^n_{p;p}(X_1, ..., X_n; Y)$$

holds true for $1 \leq p \leq q < s^* < 2$.

As we will see in the next results, a far-reaching version (of independent interest) of this theorem is valid. This result uses arguments from [2] and, in essence is contained in [2]:

Theorem 4.2. *If X_1 has cotype 2 and $1 \leq p \leq s \leq 2$, then*

$$\Pi^n_{(s;s,t,...,t)}(X_1, ..., X_n; Z) \subset \Pi^n_{(p;p,t,...,t)}(X_1, ..., X_n; Z)$$

for all $X_2, ..., X_n, Z$ and all $t \geq 1$. In particular

$$\Pi^n_{(s;s,...,s)}(X_1, ..., X_n; Z) \subset \Pi^n_{(p;p,s,...,s)}(X_1, ..., X_n; Z) \subset \Pi^n_{(p;p,p,...,p)}(X_1, ..., X_n; Z). \tag{4.1}$$

Proof. Since X_1 has cotype 2, then, using results from [2], we have

$$\ell^w_p(X_1) = \ell_r \ell^w_s(X_1)$$

for

$$\frac{1}{r} + \frac{1}{s} = \frac{1}{p}.$$

Let $(x^{(1)}_k)^\infty_{k=1} \in \ell^w_p(X_1)$ and $(x^{(i)}_k)^\infty_{k=1} \in \ell^w_t(X_i)$ for $i = 2, ..., n$. So $x^{(1)}_k = \alpha_k y_k$, with $(\alpha_k)^\infty_{k=1} \in \ell_r$ and $(y_k)^\infty_{k=1} \in \ell_s$ for all k. If $A \in \Pi^n_{(s;s,t,...,t)}(X_1, ..., X_n; Z)$, then

$$\begin{aligned}
\left(\sum_{j=1}^{\infty} \left\| A(x^{(1)}_j, ..., x^{(n)}_j \right\|^p \right)^{1/p} &= \left(\sum_{j=1}^{\infty} \left\| \alpha_j A(y_j, x^{(2)}_j, ..., x^{(n)}_j \right\|^p \right)^{1/p} \\
&\leq \left(\sum_{j=1}^{\infty} |\alpha_j|^r \right)^{1/r} \left(\sum_{j=1}^{\infty} \left\| A(y_j, x^{(2)}_j, ..., x^{(n)}_j \right\|^s \right)^{1/s} < \infty
\end{aligned}$$

and the proof is done. □

Remark 4.3. *Note that using the inclusion theorem for absolutely summing multilinear operators and Theorem 4.2 we conclude that, in fact,*

$$\Pi^n_{(s;s,...,s)}(X_1, ..., X_n; Z) = \Pi^n_{(p;p,s,...,s)}(X_1, ..., X_n; Z)$$

under the hypotheses of Theorem 4.2.

A similar result holds for spaces with cotype greater than 2:

Theorem 4.4. *If X_1 has cotype $s > 2$ and $1 \leq p \leq q < s^*$, then*

$$\Pi^n_{(q;q,t,...,t)}(X_1, ..., X_n; Z) \subset \Pi^n_{(p;p,t,...,t)}(X_1, ..., X_n; Z)$$

for all $X_2, ..., X_n, Z$ and all $t \geq 1$. In particular

$$\Pi^n_{(q;q,...,q)}(X_1, ..., X_n; Z) \subset \Pi^n_{(p;p,q,...,q)}(X_1, ..., X_n; Z) \subset \Pi^n_{(p;p,p,...,p)}(X_1, ..., X_n; Z).$$

Proof. Since X_1 has cotype s, then

$$\ell_p^w(X_1) = \ell_r \ell_q^w(X_1)$$

whenever $1 \leq p \leq q < s^*$ with

$$\frac{1}{r} + \frac{1}{q} = \frac{1}{p}$$

and the proof follows the lines of the proof of Theorem 4.2. □

Remark 4.5. *Obviously, Theorems 4.2 and 4.4 have an analogous version when some X_j (instead of necessarily X_1) has cotype* 2.

The following result can be found in [31, Theorem 3.10]:

Proposition 4.6 (Pérez-García and Villanueva, 2003)**.** *If Y has cotype finite cotype s, then*

$$\bigcup\nolimits_{p\geq 1} \delta_p^n(X_1,...,X_n;Y) \subset \Pi_{(s;2,...,2)}^{mult,n}(X_1,...,X_n;Y) \subset \Pi_{(s;2,...,2)}^{n}(X_1,...,X_n;Z)$$

for all Banach spaces $X_1,...,X_n,Z$.

In particular, the previous result shows that

$$\mathcal{L}(Y;Z) \circ \bigcup\nolimits_{p\geq 1} \delta_p^n(X_1,...,X_n;Y) \subset \Pi_{(s;2,...,2)}^{n}(X_1,...,X_n;Z)$$

for all Banach spaces $X_1,...,X_n,Z$ and Y with finite cotype s.

We will focus our attention in the case $s=2$ of Proposition 4.6. It is well-known that if $X_1,...,X_n,Y$ have cotype 2, then

$$\Pi_2^{\mathrm{mult},n}(X_1,...,X_n;Y) = \Pi_r^{\mathrm{mult},n}(X_1,...,X_n;Y)$$

for every $1 \leq r \leq 2$. Hence

Corollary 4.7. *If $X_1,...,X_n,Y$ have cotype* 2*, then*

$$\Pi_q(Y;Z) \circ \bigcup\nolimits_{p\geq 1} \delta_p^n(X_1,...,X_n;Y) \subset \Pi_r^{mult,n}(X_1,...,X_n;Y)$$

for all $q \in [1,\infty), 1 \leq r \leq 2$ and all Banach space Z.

Under the assumptions of Proposition 4.6 in general $\Pi_1^n(X_1,...,X_n;Z)$ is not contained in $\Pi_2^n(X_1,...,X_n;Z)$. The map $T : \ell_2 \times \ell_2 \to \ell_1$ given by $T(x,y) = (x_j y_j)_{j=1}^{\infty}$ belongs to $\Pi_1^2(\ell_2,\ell_2;\ell_1)$ but not to $\Pi_2^2(\ell_2,\ell_2;\ell_1)$. In fact Theorem 4.2 (and Remark 4.5), in particular, assures that if some X_j has cotype 2, then

$$\Pi_{(2;2,2,...,2)}^{n}(X_1,...,X_n;Z) \subset \Pi_{(p;p,p,...,p)}^{n}(X_1,...,X_n;Z)$$

for all $1 \leq p \leq 2$. So we have:

Corollary 4.8. *If X_j has cotype* 2 *for some j and Y has cotype* 2 *then*

$$\Pi_q(Y;Z) \circ \bigcup\nolimits_{p\geq 1} \delta_p^n(X_1,...,X_n;Y) \subset \Pi_r^{n}(X_1,...,X_n;Z)$$

for all $q \in [1,\infty)$, all $r \in [1,2]$ and all Banach spaces $X_1,...,X_{j-1},X_{j+1},...,X_n,Z$.

Hence Corollary 4.8 is quite stronger from Theorem 1.3 for this special case where X_j has cotype 2 for some j and Y has cotype 2.

Now we explore some consequences of Proposition 3.2. Note that if Y and Z have cotype 2, for example, it is well-known that $\Pi_1(Y,Z) = \Pi_q(Y,Z)$ for all $1 \leq q < \infty$ (see [17, Corollary 11.16]) and we get:

Corollary 4.9. *If Y and Z have cotype 2, then*

$$\Pi_q(Y;Z) \circ \bigcup_{p\geq 1} \delta_p^n(X_1,...,X_n;Y) \subset \Pi^n_{(1;2,...,2)}(X_1,...,X_n;Z) \subset \Pi^n_{(2;2,...,2)}(X_1,...,X_n;Z)$$

for all $q \in [1,\infty)$ and all Banach spaces $X_1,...,X_n$.

Also, using [17, Corollary 11.16] and Proposition 3.2 we have:

Corollary 4.10. *If Y has cotype 2, then*

$$\Pi_q(Y;Z) \circ \bigcup_{p\geq 1} \delta_p^n(X_1,...,X_n;Y) \subset \Pi^n_{(1;2,...,2)}(X_1,...,X_n;Z) \subset \Pi^n_{(2;2,...,2)}(X_1,...,X_n;Z)$$

for all $q \in [1,2]$ and all Banach spaces $X_1,...,X_n,Z$.

5. A partial solution to Problem 1.4

In this last section we present some very simple but apparently useful and overlooked remarks on Problem 1.4. It is easy to see that

$$\frac{p}{n} \leq r \leq p \Rightarrow \delta_p^n \subset \Pi_r^n. \tag{5.1}$$

We thus have:

Proposition 5.1. *Let $p,q,r \in (1,\infty)$ be such that $r \leq p$. Then the inclusion*

$$\Pi_q \circ \delta_p^n \subset \Pi_r^n \tag{5.2}$$

is valid for all $n \geq \frac{p}{r}$.

So, *a fortiori*, we have a partial answer to Problem 1.4 (note that we do not actually need the hypothesis $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$):

Corollary 5.2. *Let $p,q,r \in (1,\infty)$ be such that $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. Then the inclusion*

$$\Pi_q \circ \delta_p^n \subset \Pi_r^n \tag{5.3}$$

is valid for all $n \geq \frac{p}{r}$.

References

[1] R. Alencar and M. C. Matos, Some classes of multilinear mappings between Banach spaces, Publicaciones del Departamento de Análisis Matemático 12, Universidad Complutense Madrid, (1989).

[2] O. Blasco, G. Botelho, D. Pellegrino and P. Rueda, Lifting summability properties for multilinear mappings, preprint.

[3] G. Botelho, Cotype and absolutely summing multilinear mappings and homogeneous polynomials, Proc. Roy. Irish Acad Sect. A **97** (1997), 145-153.

[4] G. Botelho, Ideals of polynomials generated by weakly compact operators, Note Mat. **25** (2005/2006), 69-102.

[5] G. Botelho, C. Michels and D. Pellegrino, Complex interpolation and summability properties of multilinear operators, Rev. Matem. Complut. **23** (2010), 139-161.
[6] G. Botelho and D. Pellegrino, Dominated polynomials in $\mathcal{L}_p$ spaces, Arch. Math. **83** (2004), 364-370.
[7] G. Botelho and D. Pellegrino, Scalar-valued dominated polynomials on Banach spaces, Proc. Amer. Math. Soc. **134** (2006), 1743-1751.
[8] G. Botelho and D. Pellegrino, Coincidence situations for absolutely summig non-linear mappings, Port. Math. **64** (2007), 175-191.
[9] G. Botelho and D. Pellegrino, Absolutely summing operators into spaces with no finite cotype. Bulletin of the Belgian Mathematical Society Simon Stevin **16** (2009), 373-378.
[10] G. Botelho, D. Pellegrino and P. Rueda, On composition ideals of multilinear mappings and homogeneous polynomials, Publ. Res. Ins. Math. Sci. **43** (2007), 1139–1155.
[11] G. Botelho, D. Pellegrino and P. Rueda, Dominated polynomials on infinite-dimensional spaces, Proc. Amer. Math. Soc. **138** (2010), 209-216.
[12] G. Botelho, D. Pellegrino and P. Rueda, Cotype and absolutely summing linear operators, Mathematische Zeitschrift **267** (2011), 1-7.
[13] E. Çalışkan and D. M. Pellegrino, On the multilinear generalizations of the concept of absolutely summing operators, Rocky Mount. J. Math. **37** (2007), 1137-1154.
[14] D. Carando, V. Dimant and S. Muro, Coherent sequences of polinomial ideals on Banach spaces. Math. Nachr. 282 (2009), 1111-1133.
[15] D. Carando, V. Dimant and S. Muro, Every Banach ideal of polynomials is compatible with an operator ideal, to appear in Monatshefte für Mathematik, DOI: 10.1007/s00605-010-0255-3.
[16] R. Cilia and J. Gutiérrez, Dominated, diagonal polynomials on ℓ_p spaces, Arch. Math. **84** (2005), 421-431.
[17] J. Diestel, H. Jarchow and A. Tonge, Absolutely summing operators, Cambridge University Press, 1995.
[18] J. Diestel, H. Jarchow and A. Pietsch, Operator ideals, in Handbook of the geometry of Banach spaces, Vol. I, 437-496, North-Holland, Amsterdam, 2001.
[19] V. Dimant, Strongly p-summing multilinear operators, J. Math. Anal. Appl. **278** (2003), 182-193.
[20] K. Floret and M.C. Matos, Applications of a Khinchine inequality to holomorphic mappings, Math. Nachr. **176** (1995), 65-72.
[21] H. Junek, M.C. Matos and D. Pellegrino, Inclusion theorems for absolutely summing holomorphic mappings, Proc. Amer. Math. Soc. **136** (2008), 3983-3991.
[22] T. Kühn and M. Mastyło, Products of operator ideals and extensions of Schatten classes, Math. Nachr. **283** (2010), 891-901.
[23] T. Kühn and M. Mastyło, Weyl numbers and eigenvalues of abstract summing operators, J. Math. Anal. Appl. **369** (2010), 408-422.
[24] M. Matos, Absolutely summing holomorphic mappings, Anais da Academia Brasileira de Ciências **68** (1996), 1-13.
[25] M.C. Matos, Fully absolutely summing and Hilbert-Schmidt multilinear mappings, Collect. Math. **54** (2003), 111-136.
[26] D. Pellegrino, Cotype and absolutely summing homogeneous polynomials in $\mathcal{L}_p$ spaces, Studia Math. **157** (2003), 121-131.
[27] D. Pellegrino, Almost summing mappings, Arch. Math. **82** (2004), 68-80.
[28] D. Pellegrino and J. Santos, Absolutely summing operators: a panorama, to appear in Quaest. Math.
[29] D. Pérez-García, Operadores multilineales absolutamente sumantes, Dissertation, Universidad Complutense de Madrid, 2002.
[30] D. Pérez-García, Comparing different classes of absolutely summing multilinear operators, Arch. Math. **85** (2005), 258-267.
[31] D. Pérez-García and I. Villanueva, Multiple summing operators on Banach spaces, J. Math. Anal. Appl. **285** (2003), 86-96.

[32] A. Pietsch, Absolut p-summierende Abbildungen in normieten Räumen, Studia Math. **27** (1967), 333-353.
[33] A. Pietsch, Operator Ideals, Deutscher Verlag der Wiss, 1978 and North Holland, Amsterdam, 1980.
[34] A. Pietsch, Ideals of multilinear functionals, Proceedings of the Second International Conference on Operator Algebras, Ideals and Their Applications in Theoretical Physics, 185-199, Teubner-Texte, Leipzig, 1983.
[35] D. Popa, Multilinear variants of Maurey and Pietsch theorems and applications, J. Math. Anal. Appl. **368** (2010) 157–168.
[36] D. Popa, Multilinear variants of Pietsch's composition theorem, J. Math. Anal. Appl. **370** (2010), 415-430.
[37] D. Puglisi and J. B. Seoane-Sepúlveda, Bounded linear non-absolutely summing operators, J. Math. Anal. Appl. 338 (2008), 292-298.
[38] M. Talagrand, Cotype and $(q, 1)$-summing norms in Banach spaces, Invent. Math. **110** (1992), 545-556.

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